Formulas for tune shift and β beat due to perturbations in circular accelerators

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(Received 23 September 2004; published 28 March 2005)

Rigorous formulas for nonlinear tune shift and β-function distortion due to perturbations in the focusing forces are presented, which complement the well-known tune-shift formula for quadrupole errors. Using these formulas, the calculation of nonlinear chromaticity given by Takao *et al.* [Phys. Rev. E **70**, 016501 (2004)] can be greatly simplified and extended to higher order. In addition, an expression for the nonlinear chromatic β-function distortion is given.

DOI: 10.1103/PhysRevE.71.036502 PACS number(s): 41.85.−p, 29.27.Bd

I. INTRODUCTION

A basic task in circular accelerator design and operation is to understand and control the variations of important lattice properties such as the tune and β function under the influence of small perturbations in focusing forces due to field gradient errors, chromatic effects, and so on. Many perturbative treatments exist in the literature. Recently, in the context of nonlinear chromaticity, Takao *et al.* pointed out that there is a lack of rigorous results especially for high-order effects, and in fact, many treatments are inaccurate or simply wrong [1]. They then provided a rigorous derivation for up to third-order chromaticity. Although we share their view on the general lack of rigorousness in many of the treatments, rigorous perturbative theory does exist. In fact, an accurate formula for tune calculation traces back to Hill's century-old original work [2]. In this paper, we revise the results on the perturbative computation of tune and β function and present simple practical formulas for computing the nonlinear tune shift and β-function distortion due to gradient errors up to third and second order, respectively. Then we show that Takao *et al.*'s results on nonlinear chromaticity can be obtained much more easily from our tune-shift formula, and as an extension, we present the expression for the nonlinear chromatic β-beat factor. This work is stimulated by the work of Takao *et al.* and results from an intended comment on their work. In addition to extending [1], we hope to raise awareness of the rigorous perturbation theory and Hill's original result as well as proper formulas for the tune shift and β-function distortion, which complement the well-known tune-shift formula for quadrupole errors, Eq. (13) below.

II. TUNE SHIFT AND β-FUNCTION DISTORTION DUE TO GRADIENT ERRORS

Let us start with the Hamiltonian

$$H(x,p;s) = \frac{p^2}{2} + (K_0 + \Delta K)\frac{x^2}{2}, \tag{1}$$

where $K_0(s)$ is the unperturbed focusing strength and ΔK is the perturbation. A general problem is to solve the perturbed system knowing the solution of the unperturbed one whose one-turn phase advance and Courant-Snyder parameters are μ_0, β_0, and α_0. In order to simplify the problem and highlight the perturbation, it is convenient to use the normal coordinates of the unperturbed system $\overline{x}=x/\sqrt{\beta_0}$ and $\overline{p}=(\beta_0 p - \alpha_0 x)/\sqrt{\beta_0}$. Using the generating function $F_2=(1/\sqrt{\beta_0})x\overline{p} - (\alpha_0/\beta_0)x^2/2$ and the condition $\alpha_0'=K_0\beta_0-\gamma_0$, the new Hamiltonian becomes

$$\overline{H}(\overline{x},\overline{p};s) = \frac{\overline{p}^2}{2\beta_0} + (1+\beta_0^2\Delta K)\frac{\overline{x}^2}{2\beta_0}. \tag{2}$$

Furthermore, using the unperturbed phase advance $\psi(s) = \int^s ds'/\beta_0(s')$ instead of s as the time variable, the Hamiltonian simply reads

$$\overline{H}(\overline{x},\overline{p};\psi) = \frac{\overline{p}^2}{2} + (1+\beta_0^2\Delta K)\frac{\overline{x}^2}{2}. \tag{3}$$

Let μ, β, and α be the one-turn phase advance and Courant-Snyder parameters of the perturbed system in Eq. (1), and let $\overline{\mu}$, $\overline{\beta}$, and $\overline{\alpha}$ be the corresponding parameters of the system in Eq. (3). It is not difficult to see that

$$\mu = \overline{\mu}, \quad \beta = \beta_0\overline{\beta}, \quad \alpha = \alpha_0\overline{\beta} + \overline{\alpha}. \tag{4}$$

The Hamiltonian of Eq. (3) is well suited for perturbative treatment when ΔK is small. The equation of motion is a Hill's equation that reads

$$\frac{d^2\overline{x}}{d\psi^2} + (1+\beta_0^2\Delta K)\overline{x} = 0. \tag{5}$$

The periodic focusing is naturally characterized by its Fourier coefficients ϑ_n as[1]

$$\left(\frac{\mu_0}{\pi}\right)^2(1+\beta_0^2\Delta K) = \sum_{n=-\infty}^{\infty}\vartheta_n e^{i2\pi n\psi/\mu_0}, \tag{6}$$

where the coefficient

[1]With the factor $(\mu_0/\pi)^2$, ϑ_n are effectively the Fourier coefficients for the system whose period is normalized from μ_0 to π, to be consistent with well-established mathematical results.

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$$\vartheta_0 = \left(\frac{\mu_0}{\pi}\right)^2 \left(1 + \frac{1}{\mu_0}\oint ds\, \beta_0 \Delta K\right) \tag{7}$$

is the average focusing and the other coefficients are given by

$$\vartheta_n = \frac{\mu_0}{\pi^2}\oint ds\, e^{-i2\pi n\psi/\mu_0}\beta_0\Delta K \quad (n \neq 0). \tag{8}$$

Using the Fourier coefficients, an exact expression for the phase advance μ was obtained originally by Hill and others in terms of an infinite determinant, known as Hill's determinant, as [2–6]

$$\cos\mu = 1 - 2\sin^2\left(\frac{\pi}{2}\sqrt{\vartheta_0}\right)D. \tag{9}$$

Here the infinite determinant D is given by

$$D = \left\| \delta_{nm} + \frac{\tilde{\vartheta}_{n-m}}{\vartheta_0 - (2n)^2} \right\|_{-\infty}^{\infty}, \tag{10}$$

where $\tilde{\vartheta}_0 = 0$ and $\tilde{\vartheta}_n = \vartheta_n$ for $n \neq 0$. Expanding D up to the third order in ϑ_n yields [3,4]

$$\cos\mu \simeq \cos(\sqrt{\vartheta_0}\pi) + \frac{\pi\sin\sqrt{\vartheta_0}\pi}{4\sqrt{\vartheta_0}}\sum_{n=1}^{\infty}\frac{|\vartheta_n|^2}{\vartheta_0 - n^2} + \frac{\pi\sin\sqrt{\vartheta_0}\pi}{8\sqrt{\vartheta_0}} \times \sum_{m,n=1}^{\infty}\frac{\mathrm{Re}(\vartheta_m\vartheta_n\vartheta_{m+n}^*)(m^2+n^2+mn-3\vartheta_0)}{(\vartheta_0-m^2)(\vartheta_0-n^2)[\vartheta_0-(m+n)^2]}. \tag{11}$$

Since it is often important to know the perturbative phase shift $\Delta\mu \equiv \mu - \mu_0$, we expand it into a series accurate to the third order in ϑ_n as

$$\Delta\mu = \frac{\mu_0 u}{2} - \frac{\mu_0 u^2}{8} - \frac{v}{\sin\mu_0} + \frac{\mu_0 u^3}{16} + \frac{\mu_0 \cos\mu_0\, u\, v}{2\sin^2\mu_0}, \tag{12}$$

where the two small quantities u and v are defined via Eq. (11) and the equation $\cos\mu = \cos(\mu_0\sqrt{1+u}) + v$, i.e., $u = (1/\mu_0)\oint ds\, \beta_0\Delta K$, which is the average error, and v is the last two terms of Eq. (11). Note that u and v are at least first- and second-order quantities, respectively. The well-known tune-shift expression [7]

$$\Delta\nu = \frac{\Delta\mu}{2\pi} = \frac{1}{4\pi}\oint ds\, \beta_0\Delta K \tag{13}$$

is the first-order term in Eq. (12).

The β function has been derived recently in [3] to the second order in ϑ_n as

$$\overline{\beta} \simeq \frac{\mu_0}{\pi}\frac{\sin(\sqrt{\vartheta_0}\pi)}{\sqrt{\vartheta_0}\sin\mu}(1+w), \tag{14}$$

where

$$w = \sum_{n=1}^{\infty}\frac{\mathrm{Re}[\vartheta_n e^{i2n\pi\psi/\mu_0}]}{n^2-\vartheta_0} + \frac{\pi\cot(\sqrt{\vartheta_0}\pi)}{4\sqrt{\vartheta_0}}\sum_{n=1}^{\infty}\frac{|\vartheta_n|^2}{n^2-\vartheta_0} + \sum_{m,n=1}^{\infty}\left\{\frac{(m^2+mn+n^2-3\vartheta_0)\mathrm{Re}[\vartheta_m\vartheta_n e^{i2(m+n)\pi\psi/\mu_0}]}{4(m^2-\vartheta_0)(n^2-\vartheta_0)[(m+n)^2-\vartheta_0]} + \frac{(m^2-mn+n^2-3\vartheta_0)\mathrm{Re}[\vartheta_m\vartheta_n^* e^{i2(m-n)\pi\psi/\mu_0}]}{4(m^2-\vartheta_0)(n^2-\vartheta_0)[(m-n)^2-\vartheta_0]}\right\}.$$

This β-beat factor of the perturbed system can be expanded accurately to the second order as

$$\overline{\beta} = 1 - \frac{u}{2} + \frac{3u^2}{8} + \frac{\cos\mu_0\, v}{\sin^2\mu_0} + w - \frac{uw}{2}. \tag{15}$$

Note that the v term actually cancels with the second term of w. Also note that $w(s)$ gives rise to an s-dependent variation. The high-order terms of u in both Eq. (12) and Eq. (15) are usually negligible. The nonlinear behavior is mostly due to the resonant terms in v and w.

III. NONLINEAR CHROMATICITY

The chromatic tune shift $\Delta\mu/2\pi$ due to momentum deviation $\delta = \Delta p/p$ is parametrized by the linear and nonlinear chromaticity ξ_i as

$$\frac{\Delta\mu}{2\pi} = \xi_1\delta + \xi_2\delta^2 + \xi_3\delta^3 + \cdots. \tag{16}$$

The chromaticity can be readily derived using the tune-shift formula in Eq. (12) and the perturbative chromatic focusing described in [1], i.e.,

$$\Delta K = G(s,\delta) = \sum_n G_n\delta^n. \tag{17}$$

To distinguish the Fourier coefficients of G_1 and G_2, left subscripts ${}_1\vartheta_n$ and ${}_2\vartheta_n$ will be used.

The linear and quadratic chromaticity are obvious from the first term and the first three terms of Eq. (12), respectively,

$$\xi_1 = \frac{1}{4\pi}\overline{\beta_0 G_1}, \tag{18}$$

$$\xi_2 = \frac{1}{4\pi}\overline{\beta_0 G_2} - \frac{1}{16\pi\mu_0}\overline{\beta_0 G_1}^{\,2} - \frac{\pi^3}{8\mu_0}\sum_{n=1}^{\infty}\frac{|{}_1\vartheta_n|^2}{\mu_0^2 - \pi^2 n^2}. \tag{19}$$

The overbar is a shorthand for the integral $\oint ds(\cdots)$. The cubic chromaticity is straight-forward to compute, paying attention to all the ways to generate third-order terms [8]. The result reads

$$
\begin{aligned}
\xi_3 = {} & \frac{1}{4\pi}\overline{\beta_0 G_3} - \frac{1}{8\pi\mu_0}\overline{\beta_0 G_1}\,\overline{\beta_0 G_2} \\
& - \frac{\pi^3}{4\mu_0}\sum_{n=1}^{\infty}\frac{\mathrm{Re}\left({}_1\vartheta_n\,{}_2\vartheta_n^*\right)}{\mu_0^2-\pi^2 n^2} + \frac{1}{32\pi\mu_0^2}\overline{\beta_0 G_1}^{\,3} \\
& + \frac{\pi^3}{16\mu_0^2}\overline{\beta_0 G_1}\sum_{n=1}^{\infty}\frac{(3\mu_0^2-\pi^2 n^2)\left|{}_1\vartheta_n\right|^2}{(\mu_0^2-\pi^2 n^2)^2} + \frac{\pi^5}{16\mu_0} \\
& \times \sum_{m,n=1}^{\infty}\frac{\mathrm{Re}\left({}_1\vartheta_m\,{}_1\vartheta_n\,{}_1\vartheta_{m+n}^*\right)[3\mu_0^2-\pi^2(m^2+n^2+mn)]}{(\mu_0^2-\pi^2 m^2)(\mu_0^2-\pi^2 n^2)[\mu_0^2-\pi^2(m+n)^2]}.
\end{aligned}
\tag{20}
$$

These are the same results as obtained in [1] through a much lengthier derivation, given the relations $\oint ds\, \beta_0 G_m = (\mu_0/2)a_m(0)$ and ${}_m\vartheta_n = [\mu_0^2/(2\pi^2)][a_m(n) - i\, b_m(n)]$ for nonzero n.

IV. NONLINEAR CHROMATIC β-FUNCTION DISTORTION

The chromatic β-function distortion can be parametrized as

$$
\frac{\beta}{\beta_0} = \overline{\beta} = 1 + \beta_1\delta + \beta_2\delta^2 + \cdots . \tag{21}
$$

Using the β-beat formula in Eq. (15), the linear and quadratic coefficients can be written as

$$
\beta_1 = -\frac{1}{2\mu_0}\overline{\beta_0 G_1} - \sum_{n=1}^{\infty}\frac{\pi^2\mathrm{Re}\left[{}_1\vartheta_n e^{i2\pi n\psi/\mu_0}\right]}{\mu_0^2-\pi^2 n^2}, \tag{22}
$$

$$
\begin{aligned}
\beta_2 = {} & -\frac{1}{2\mu_0}\overline{\beta_0 G_2} - \sum_{n=1}^{\infty}\frac{\pi^2\mathrm{Re}\left[{}_2\vartheta_n e^{i2\pi n\psi/\mu_0}\right]}{\mu_0^2-\pi^2 n^2} + \frac{3}{8\mu_0^2}\overline{\beta_0 G_1}^{\,2} \\
& + \frac{1}{2\mu_0}\overline{\beta_0 G_1}\sum_{n=1}^{\infty}\frac{\pi^2\mathrm{Re}\left[{}_1\vartheta_n e^{i2\pi n\psi/\mu_0}\right]}{\mu_0^2-\pi^2 n^2} \\
& + \sum_{m,n=1}^{\infty}\left\{\frac{\pi^4(\pi^2 k_+ - 3\mu_0^2)\mathrm{Re}\left[{}_1\vartheta_m\,{}_1\vartheta_n e^{i2(m+n)\pi\psi/\mu_0}\right]}{4(\pi^2 m^2-\mu_0^2)(\pi^2 n^2-\mu_0^2)[\pi^2(m+n)^2-\mu_0^2]}\right. \\
& \qquad \left. + \frac{\pi^4(\pi^2 k_- - 3\mu_0^2)\mathrm{Re}\left[{}_1\vartheta_m\,{}_1\vartheta_n^* e^{i2(m-n)\pi\psi/\mu_0}\right]}{4(\pi^2 m^2-\mu_0^2)(\pi^2 n^2-\mu_0^2)[\pi^2(m-n)^2-\mu_0^2]}\right\},
\end{aligned}
\tag{23}
$$

where $k_\pm = m^2 \pm mn + n^2$ to shorten the expression.

V. CONCLUDING REMARKS

We presented a rigorous perturbation theory for computing nonlinear tune shift and β beating due to field gradient errors and chromatic effects. Practical formulas up to third order for tune shift and second order for beta beating are given. Even higher-order tune shifts can be obtained via Hill's determinant. Using the tune-shift formula, Takao *et al.*'s derivation of the nonlinear chromaticity is greatly simplified. As an important complement, the expression for nonlinear chromatic β distortion is given. These formulas provide direct connections between the lattice properties and the Fourier coefficients of the focusing field, and thus should be useful for lattice analysis and optimization (of chromatic tune shift and/or β distortion at the interaction points, for example).

ACKNOWLEDGMENTS

I would like to thank L. C. Teng for commenting on the manuscript. This work was supported by the U.S. Department of Energy, Office of Basic Energy Sciences, under Contract No. W-31-109-ENG-38.

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[8] First note that $\Delta\mu_3$ can be written as

$$
\frac{1}{2}\oint ds\, \beta_0 G_3 - \frac{\mu_0}{4}u_1 u_2 + \frac{\mu_0}{16}u_1^3 - \frac{v_3}{\sin\mu_0} + \frac{\mu_0\cot\mu_0 u_1 v_2}{2\sin\mu_0}
$$

where the subscripts of u and v indicate their order in δ. The term v_3 contains an obvious term from the last term of Eq. (11) and the following terms from the second one:

$$
\begin{aligned}
& \frac{\pi^4\sin\mu_0}{4\mu_0}\sum_{n=1}^{\infty}\left[\frac{2\mathrm{Re}\left({}_1\vartheta_n\,{}_2\vartheta_n^*\right)}{\mu_0^2-\pi^2 n^2} + \frac{\mu_0^2 u_1\left|{}_1\vartheta_n\right|^2}{(\mu_0^2-\pi^2 n^2)^2}\right] \\
& \quad + (\mu_0\cot\mu_0 - 1)\frac{u_1 v_2}{2}.
\end{aligned}
$$

Note that the term $\mu_0\cot\mu_0 u_1 v_2/2$ will be canceled by the last term of the previous expression.